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EXISTNCE OF QUASIISOMETRIC MAPPINGS AND ROYDEN COMPACTIFICATIONS [1]

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1. Introduction. Consider a d-dimensional ($d \geq 2$) Riemannian manifold D of class C^{∞} which is orientable and countable but not necessarily connected and given an exponent $1 < p < \infty$. The *Royden p-algebra* $M_p(D)$ of D is defined by $M_p(D) := L^{1,p}(D) \cap L^{\infty}(D) \cap C(D)$, which is a commutative Banach algeba, i.e. the so-called normed ring, under pointwise addition and multiplication with $\|u; M_p(D)\| := \|u; L^{\infty}(D)\| + \|\nabla u; L^p(D)\|$ as norm, where $L^{1,p}(D)$ is the Dirichlet space, i.e. the space of locally integrable real valued functions u on D whose distributional gradients ∇u of u belong to $L^p(D)$ considered with respect to the metric structure on D. The maximal ideal space D_p^* (cf. e.g. p.298 in [20]) of $M_p(D)$ is referred to as the *Royden p-compactification* of D, which is also characterized as the compact Hausdorff space containing D as its open and dense subspace such that every function in $M_p(D)$ is continuously extended to D_p^* and $M_p(D)$ is uniformly dense in $C(D_p^*)$ (cf. e.g. [17], [18], [11] and also p.154 in [14]).

Suppose that D and D' are d-dimensional ($d \geq 2$) Riemannian manifolds of class C^{∞} which are orientabl and countable but not necessarily connected. Moreover we always assume in this note that none of the components of D and D' is compact, which is however not an essential restriction and postulated only for the sake of simplicity. In 1982, the present author and H. Tanaka [13] (see also [10]) jointly showed that two conformal Royden compactifications D_d^* and $(D')_d^*$ are homeomorphic if and only if there exists an almost quasiconformal mapping of D onto D'. Here we say that a homeomorphism f of D onto D' is an *almost quasiconformal mapping* of D onto D' if there exists a compact subset $E \subset D$ such that $f = f|D \setminus E$ is a quasiconformal mapping of $D \setminus E$ onto $D' \setminus f(E)$. There are many ways of defining quasiconfrmality but the following metric defiition is convenient for applying to Riemannian manifolds (cf. e.g. p.113 in [19]): the homemrphism f of $D \setminus E$ onto $D' \setminus f(E)$ is *quasiconfomal*, by defintion, if

$$(2) \qquad \sup_{x \in D \setminus E} \left(\limsup_{r \downarrow 0} \frac{\max_{\rho(x,y)=r} \rho'(f(x), f(y))}{\min_{\rho(x,y)=r} \rho'(f(x), f(y))} \right) < \infty,$$

where ρ and ρ' are geodesic distances on $D \setminus E$ and $D' \setminus f(E)$. It has been an open question for a long period since the above result was obtained as for what can be said about the

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counterpart of the above result for nonconformal case, i.e. if the exponent d in the above result is replaced by $1 < p < d$. The *purpose* of this note is to settle this question by establishing the main theorem mentioned below.

To state our result we need to introduce a class of special kind of almost quasiconformal mappings. A homeomorphism f of D onto D' is said to be an *almost quasiisometric mapping* of D onto D' if there exists a compact set $E \subset D$ such that $f = f|D \setminus E$ is a quasiisometric mapping of $D \setminus E$ onto $D' \setminus f(E)$. Here the homeomorphism f of $D \setminus E$ onto $D' \setminus f(E)$ is *quasiisometric*, by definition, if there exists a constant $K \in [1, \infty)$ such that

$$\frac{1}{K}\rho(x,y) \leq \rho'(f(x), f(y)) \leq K\rho(x,y) \tag{3}$$

for every pair of points x and y in $D \setminus E$, where we always set $\rho(x,y) = \rho'(f(x), f(y)) = \infty$ if the component of $D \setminus E$ containing x and that containing y are different. From (3) it follows that

$$\frac{1}{K}r \leq \min_{\rho(x,y)=r} \rho'(f(x), f(y)) \leq \max_{\rho(x,y)=r} \rho'(f(x), f(y)) \leq Kr$$

for any fixed $x \in D$ and for any sufficiently small positive number $r > 0$, which implies that the left hand side term of (2) is dominated by K^2. Thus a quasiisometric mapping is automatically a quasiconformal mapping but obviously there exists a quasiconformal mapping which is not a quasiisometric mapping. Then our main result of this paper is stated as follows.

4. MAIN THEOREM. *When $1 < p < d$, Royden compactifications D_p^* and $(D')_p^*$ are homeomorphic if and only if there exists an almost quasiisometric mapping of D onto D'. More precisely, any almost quasiisometric mapping of D onto D' is uniquely extended to a homeomorphism of D_p^* onto $(D')_p^*$; conversely, the restriction to D of any homeomorphism of D_p^* onto $(D')_p^*$ is an almost quasiisometric mapping of D onto D'.*

It may be interesting to compare the above topological result with the former relevant algebraic results obtained by the present author [8] and [9], Lewis [6], and Lelon-Ferrand [5] (cf. also Soderborg [15]): Royden algebas $M_d(D)$ and $M_d(D')$ are algebraically isomorphic if and only if there exists a quasiconformal mapping of D onto D'; when $1 < p < d$, $M_p(D)$ and $M_p(D')$ are algebraically isomorphic if and only if there exists a quasiisometric mapping of D onto D'. All these results including our present main theorem are shown to be invalid when $d < p < \infty$ by giving a counter example, which will be discussed elsewhere. Another important problem related to the above main result is the following: does the existence of an almost quasiisometric (almost quasiconformal, resp.) mapping of D onto D' imply that of a quasiisometric (quasiconformal, resp) mapping of D onto D'? It is affirmative for the quasiconformal case if D is the unit ball in the d-dimensional Euclidean space $\mathbf{R}^d$ (Gehring [2], see also Soderborg [16]); it is also affirmative again for the quasiconformal case if the dimensions of D and D' are 2. Except for these partial results though not easy to prove, the problem is widely open.

5. Royden compactifications of Riemannian manifolds. By a *Riemannian manifold* D of dimension $d \geq 2$ we always mean in this note an orientable and countable but not necessarily connedted C^∞ manifold D of dimension d with a metric tensor (g_{ij}) of class C^∞. We also assume that any component of D is not compact only for the sake of simplicity.

We say that U or more precisely (U, x) is a *parametric domain* on D if the following two conditions are satisfied: firstly U is a domain, i.e. a connected open set, in D; secondly x is a C^∞ diffeomorphism of U onto a domain $x(U)$ in the Euclidean space $\mathbf{R}^d$ of dimension $d \geq 2$. The map $x = (x^1, \cdots, x^d)$ is referred to as a *parameter* on U. We often identify a generic point P of U with its parameter $x(P)$ and denote them by a same letter x, for example. In other words we view U to be embedded in $\mathbf{R}^d$ by identifying U with $x(U)$ so that U itself may be considered as a Riemannian manifold (U, g_{ij}) with metric tensor (g_{ij}) restricted on U and at the same time as an Euclidean subdomain (U, δ_{ij}) with the natural metric tensor (δ_{ij}), δ_{ij} being the Kronecker delta.

Take a parametric domain (U, x) on D. The metric tensor (g_{ij}) on D gives rise to a $d \times d$ matrix $(g_{ij}(x))$ of functions $g_{ij}(x)$ on U. We say that (U, x) is a λ-*domain* with $\lambda \in [1, \infty)$ if the following matrix inequalities hold:

$$\frac{1}{\lambda}(\delta_{i,j}) \leq (g_{ij}(x)) \leq \lambda(\delta_{ij}) \tag{6}$$

for every $x \in U$. It is important that any point of D has a λ-domain as its neighborhood for any $\lambda \in (1, \infty)$. This comes from the fact that there exists a parametric ball (U, x) at any point $P \in D$ (i.e. a parametric domain (U, x) such that $x(P) = 0$ and $x(U)$ is a ball in $\mathbf{R}^d$ centered at the origin 0) such that $(g_{ij}(x))$ with respect to (U, x) satisfies $g_{ij}(0) = \delta_{ij}$.

The metric tensor (g_{ij}) on D defines the line element ds on D by $ds^2 = g_{ij}(x)dx^i dx^j$ in each parametric domain $(U, x = (x^i, \cdots, x^d))$. Here and hereafter we follow the Einstein convention: whenever an index i appears both in the upper and lower positions, it is understood that summation for $i = 1, \cdots, d$ is carried out. The length of a rectifiable curve γ on D is given by $\int_\gamma ds$. The *geodesic distance* $\rho(x, y)$ between two points x and y in D is given by

$$\rho(x, y) = \rho_D(x, y) = \inf_\gamma \int_\gamma ds,$$

where the infimum is taken with respect to rectifiable curves γ connecting x and y. Needless to say, if there is no such curve γ, i.e. if x and y are in the different components of D, then, as the infimum of empty set, we understand that $\rho(x, y) = \infty$. When (U, x) is a parametric domain and considered as the Riemannian manifold (U, δ_{ij}), then $\rho_U(x, y)$ can also be given by

$$\rho(x, y) = \rho_U(x, y) = \inf \sum_{i=0}^{n} |x_i - x_{i-1}|,$$

where the infimum is taken with respect to every polygonal line $x = x_0, x_1, \cdots, x_{n-1}, x_n = y$ such that every line segment $[x_{i-1}, x_i] = \{(1-t)x_{i-1} + tx_i : 0 \leq t \leq 1\} \subset U$ for each $i = 1, \cdots, n$.

We write $(g^{ij}) := (g_{ij})^{-1}$ and $g := \det(g_{ij})$. We denote by dV the volume elemnt on D so that

$$dV(x) = \sqrt{g(x)}dx^1 \wedge \cdots \wedge dx^d$$

in each parametric domain $(U, x = (x^1, \cdots, x^d))$. On (U, δ_{ij}) we also have the volume element (Lebesgue measure) $dx = dx^1 \cdots dx^d$. Sometimes we use dx to mean $(dx^1, \cdots, dx^d)$ but there will be no confusion by context. The Riemannian volume element $dV(x)$ and the Euclidean (Lebesgue) volume element dx are mutually absolutely continuous and the Radon-Nikodym densities $dV(x)/dx = \sqrt{g(x)}$ and $dx/dV(x) = 1/\sqrt{g(x)}$ are locally bounded on U. Thus a.e.dV and a.e.dx are identical and we can loosely use a.e. without referring to dV or dx.

For each $x \in D$, the tangent space to D at x will be denoted by T_xD. We denote by $\langle h, k\rangle$ the inner product of two tangent vectors h and k in T_xD and by $|h|$ the length of $h \in T_xD$ so that if $(h_1, \cdots, h_d)$ and $(k_1, \cdots, k_d)$ are covariant components of h and k, then

$$\langle h, k\rangle = g^{ij}h_ik_j \qquad \text{and} \qquad |h| = \langle h, h\rangle^{1/2} = (g^{ij}h_ih_j)^{1/2}.$$

Since we may consider two metric tensors (g_{ij}) and $(\delta_{i,j})$ on a parametric domain (U, x), we occasionally write $\langle h, k\rangle_{g_{ij}}$ or $\langle h, k\rangle_{\delta_{ij}}$ and similarly $|h|_{g_{ij}}$ or $|h|_{\delta_{ij}}$ to make clear whether they are considered on (U, g_{ij}) or on (U, δ_{ij}).

Let G be an open subset of D. In this note we use the notation $L^p(G)$ $(1 \leq p \leq \infty)$ in two ways. The first is the standard use: $L^p(G) = L^p(G; g_{ij})$ is the Banach space of measurable functions u on G with the finite norm $\|u; L^p(G)\|$ given by

$$\|u; L^p(G)\| := \left(\int_G |u|^p dV\right)^{\frac{1}{p}} \qquad (1 \leq p < \infty)$$

and $\|u; L^\infty(G)\|$ is the essential supremum of $|u|$ on G. The second use: for a measurable vector field X on G we write $X \in L^p(G) = L^p(G; g_{ij})$ if $|X| = |X|_{g_{ij}} \in L^p(G)$ in the first sense and we set

$$\|X; L^p(G)\| := \||X|; L^p(G)\|.$$

The *Dirichlet space* $L^{1,p}(G) = L^{1,p}(G; g_{ij})$ $(1 \leq p \leq \infty)$ is the class of functions $u \in L^1_{loc}(G)$ with the distributional gradients $\nabla u \in L^p(G)$, where the distributional gradient ∇u is determined by the relation

$$\int_G \langle \nabla u, \Psi\rangle dV = -\int_G u \operatorname{div}\Psi dV$$

for every C^∞ vector field Ψ on G with compact support in G. In the parametric domain (U, x) in G we have $\nabla u = (\partial u/\partial x^1, \cdots, \partial u/\partial x^d)$. If $\Psi = (\psi_1, \cdots, \psi_d)$ in U, then

$$\operatorname{div}\Psi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}g^{ij}\psi_j).$$

The *Sobolev space* $W^{1,p}(G) = W^{1,p}(G, g_{ij})$ $(1 \le p \le \infty)$ is the Banach space $L^{1,p}(G) \cap L^p(G)$ equipped with the norm

$$\|u; W^{1,p}(G)\| := \|u; L^p(G)\| + \|\nabla u; L^p(G)\|.$$

Given a Riemannian manifold D of dimension $d \ge 2$ and given an exponent $1 < p < \infty$, the *Royden p-algebra* $M_p(D)$ is the Banach algebra $L^{1,p}(D) \cap L^\infty(D) \cap C(D)$ equipped with the norm

$$\|u; M_p(D)\| := \|u; L^\infty(D)\| + \|\nabla u; L^p(D)\|. \tag{7}$$

By the standard mollifier method we can show that the subalgebra $M_p(D) \cap C^\infty(D)$ is dense in $M_p(D)$ with respect to the norm in (7). Henc $M_p(D)$ may also be defined as the completion of $\{u \in C^\infty(D) : \|u; M_p(D)\| < \infty\}$ without appealing to the Dirichlet space. It is important that $M_p(D)$ is closed under lattice operations $\cup$ and $\cap$ given by $(u \cup v)(x) = \max(u(x), v(x))$ and $(u \cap v)(x) = \min(u(x), v(x))$ (cf. e.g. p.21 in [4]). The maximal ideal space D_p^* of $M_p(D)$ is referred to as the *Royden p-compactification*, which can also be characterized as the compact Hausdorff space containing D as its open and dense subspace such that every function $u \in M_p(D)$ is continuously extended to D_p^* and $M_p(D)$, viewed as a subspace of $C(D_p^*)$ by this continuous extension, is dense in $C(D_p^*)$ with respect to its supremum norm.

8. Capacities of rings. A *ring* R in a Riemannian manifold D is a subset R of D with the following properties: R is a subdomain of D so that R is contained in a unique component D_R of D; $D_R \setminus R$ consists of exactly two components one of which, denoted by C_1, is compact and the other of which, denoted by C_0, is noncompact. The set C_1 will be referred to as the *inner part* of $R^c := D \setminus R$ and the set $D \setminus (R \cup C_1)$ as the *outer part* of R^c. We denote by $W(R)$ the class of functions $u \in W^{1,1}_{loc}(R) \cap C(D)$ such that $u = 1$ on the inner part of R^c and $u = 0$ on the outer part of R^c which includes C_0. The *p-capacity* $\mathrm{cap}_p R$ $(1 \le p \le \infty)$ of the ring $R \subset D$ is given by

$$\mathrm{cap}_p R := \inf_{u \in W(R)} \|\nabla u; L^p(R)\|^p \tag{9}$$

for $1 \le p < \infty$ and $\mathrm{cap}_\infty R := \inf_{u \in W(R)} \|\nabla u; L^\infty(R)\|$. Note that $\mathrm{cap}_p R$ does not depend upon which Riemannian manifold D the ring R is embedded as far as the metric structure on R is unaltered. The following inequality will be essentially made use of (cf. e.g. p.32 in [4]): if $1 < p < \infty$ and if R is a ring in D and R_k $(1 \le k \le n)$ are disjoint rings contained in R each of which separates the boundary components of R, then

$$(\mathrm{cap}_p R)^{\frac{1}{1-p}} \ge \sum_{k=1}^{n} (\mathrm{cap}_p R_k)^{\frac{1}{1-p}}. \tag{10}$$

Suppose that a ring R is contained in a parametric domain (U,x) on D for which two metric structures (g_{ij}) and (δ_{ij}) can be considered. If the need occurs to indicate that $\mathrm{cap}_p R$ is considered on (U,δ_{ij}), then we write

$$\mathrm{cap}_p R = \mathrm{cap}_p(R,\delta_{ij}) = \inf_{u\in W(R)} \int_R |\nabla u(x)|^p_{\delta_{ij}}\,dx;$$

if $\mathrm{cap}_p R$ is considered on (U,g_{ij}), then we write

$$\mathrm{cap}_p R = \mathrm{cap}_p(R,g_{ij}) = \inf_{u\in W(R)} \int_R |\nabla u|^p_{g_{ij}}\,dV$$

for $1 \le p < \infty$. Similar considerations are applied to $\mathrm{cap}_\infty(R,g_{ij})$ and $\mathrm{cap}_\infty(R,\delta_{ij})$. If moreover U is a λ-domain for any $\lambda \in [1,\infty)$, then (6) implies that

$$\frac{1}{\lambda^{\frac{d+p}{2}}}\mathrm{cap}_p(R,\delta_{ij}) \le \mathrm{cap}_p(R,g_{ij}) \le \lambda^{\frac{d+p}{2}}\mathrm{cap}_p(R,\delta_{ij}). \tag{11}$$

In the case $p=\infty$, the inequality corresponding to the above takes the following form: $\lambda^{-1/2}\mathrm{cap}_\infty(R,\delta_{ij}) \le \mathrm{cap}_\infty(R,g_{ij}) \le \lambda^{1/2}\mathrm{cap}_\infty(R,\delta_{ij})$, which however will not be used in this note.

We fix a parametric domain (U,x) in D. It is possible that the parametric domain is the d-dimensional Euclidean space $\mathbf{R}^d$ itself. A ring R contained in U is said to be a *spherical ring* in (U,x) if

$$R = \{x \in U : a < |x-P| < b\}, \tag{12}$$

where $P \in U$ and a and b are positive numbers with $0 < a < b < \inf_U |x-P|$. At this point we must be careful: in the case where the above R happens to be included in another parametric domain (V,y) of D, R may not be a spherical ring in (V,y) even if R is a spherical ring in (U,x). Namely, the notion of sperical rings cannot be introduced to the general Riemannian manifold D and is strictly attached to the parametric domain in question. Let R be a spherical ring in a parametric domain (U,x) with the above expression (12). Then we have (cf. e.g. p.35 in [4])

$$\mathrm{cap}_p R = \mathrm{cap}_p(R,\delta_{ij}) = \begin{cases} \omega_d\left(\dfrac{b^q-a^q}{q}\right)^{1-p} & (1<p<\infty,\ p\ne d), \\[2ex] \omega_d\left(\log\dfrac{b}{a}\right)^{1-d} & (p=d), \end{cases} \tag{13}$$

where we have set $q=(p-d)/(p-1)$ and ω_d is the surface area of the Euclidean unit sphere S^{d-1}. In passing we state that $\mathrm{cap}_1(R,\delta_{ij}) = \omega_d a^{d-1}$ and $\mathrm{cap}_\infty(R,\delta_{ij}) = 1/(b-a)$, which are also not used in this note.

Another important ring in $\mathbf{R}^d$ which we use later is a *Teichmüller ring* R_T defined by $R_T = \mathbf{R}^d \setminus \{te_1 : t \in [-1,0] \cup [1,\infty)\}$, where e_1 is the unit vector $(1,0,\cdots,0)$ in $\mathbf{R}^d$. We set

$$(14) \qquad t_d := \mathrm{cap}_d(R_T, \delta_{ij}).$$

Finally in this section we state a separation lemma on the topology of the Royden compactification. Let $(R_n)_{n\geq 1}$ be a sequence of rings R_n in D $(n = 1,2,\cdots)$ with the following properties: $(R_n \cup C_{n1}) \cap (R_m \cup C_{m1}) = \emptyset$ for $n \neq m$, where C_{n1} is the inner part of $(R_n)^c = D \setminus R_n$; $(R_n)_{n\geq 1}$ does not accumulate in D, i.e. $\{n : E \cap (\overline{R_n} \cup C_{n1}) \neq \emptyset\}$ is a finite set for any compact set E in D. Such a sequence $(R_n)_{n\geq 1}$ will be called an *admissible sequence*, which defines two disjoint closed sets X and Y in D as follows:

$$X := \bigcup_{n=1}^{\infty} C_{n1} \qquad \text{and} \qquad Y := \bigcap_{n=1}^{\infty} (D \setminus (R_n \cup C_{n1})).$$

We denote by $\mathrm{cl}(X; D_p^*)$ the closure of X in D_p^*. Although $X \cap Y = \emptyset$ in D, $\mathrm{cl}(X; D_P^*)$ and $\mathrm{cl}(Y; D_p^*)$ may intersect on the *Royden p-boundary*

$$\Gamma_p(D) := D_p^* \setminus D.$$

Concerning to this we have the following result.

15. LEMMA. *The set* $\mathrm{cl}(\cup_{n=1}^{\infty} R_n; D_p^*)$ *for an admissible sequence* $(R_n)_{n\geq 1}$ *in* D *separates* $\mathrm{cl}(X; D_p^*)$ *and* $\mathrm{cl}(Y; D_p^*)$ *in* D_p^* *in the sense that*

$$(16) \qquad (\mathrm{cl}(X; D_P^*)) \bigcap (\mathrm{cl}(Y; D_p^*)) = \emptyset$$

if and only if

$$(17) \qquad \sum_{n=1}^{\infty} \mathrm{cap}_p R_n < \infty.$$

PROOF: First we show that (16) implies (17). By (16) the Urysohn theorem assures the existence of a function $u \in C(D_p^*)$ such that $u = 3$ on $\mathrm{cl}(X; D_p^*)$ and $u = -2$ on $\mathrm{cl}(Y; D_p^*)$. Since $M_p(D)$ is dense in $C(D_p^*)$, there is a function $v \in M_p(D)$ such that $v > 2$ on X and $v < -1$ on Y. Finally let $w = ((v \cap 1) \cup 0) \in M_p(D)$, which satisfies $w|X = 1$, $w|Y = 0$ and $0 \leq w \leq 1$ on D. Set $w_n = w$ on $R_n \cup C_{n1}$ and $w_n = 0$ on $D \setminus (R_n \cup C_{n1})$ for $n = 1,2,\cdots$. Clearly $w_n \in W(R_n)$ so that $\mathrm{cap}_p R_n \leq \|\nabla w_n; L^p(R_n)\|^p$ $(n = 1,2,\cdots)$ and $w = \sum_{n=1}^{\infty} w_n$. Since the supports of w_n in D $(n = 1,2,\cdots)$ are mutually disjoint, we see that

$$\sum_{n=1}^{\infty} \mathrm{cap}_p R_n \leq \sum_{n=1}^{\infty} \|\nabla w_n; L^p(R_n)\|^p = \|\nabla w; L^p(D)\|^p \leq \|w; M_p(D)\|^p < \infty,$$

i.e. (17) has been deduced.

Conversely, suppose that (17) is the case. We wish to derive (16) from (17). Choose a function $w_n \in W(R_n)$ such that $\|\nabla w_n; L^p(R_n)\|^p < 2\mathrm{cap}_p R_n$ for each $n = 1, 2, \cdots$. We may suppose that $0 \leq w_n \leq 1$ on D by replacing w_n with $(w_n \cap 1) \cup 0$ if necessary (see e.g. p.20 in [4]). Clearly $w := \sum_{n=1}^{\infty} w_n \in M_p(D)$ since $\|w; L^\infty(D)\| = 1$ and

$$\|\nabla w; L^p(D)\|^p = \sum_{n=1}^{\infty} \|\nabla w_n; L^p(D_n)\|^p \leq 2 \sum_{n=1}^{\infty} \mathrm{cap}_p R_n < \infty.$$

Observe that $w = 1$ on X and $w = 0$ on Y. Hence, by the continuity of w on D_p^*, we see that $w = 1$ on $\mathrm{cl}(X; D_p^*)$ and $w = 0$ on $\mathrm{cl}(Y; D_p^*)$, which yields (16). □

As a consequence of the separation lemma above we can characterize points in the Royden p-boundary $\Gamma_p(D) = D_p^* \setminus D$ among points in D_p^* in terms of their being not G_δ for $1 \leq p \leq d$. This is no longer true for $d < p \leq \infty$. Recall that a point $\zeta \in D_p^*$ is said to be G_δ if there exists a countable sequence $(\Omega_i)_{i \geq 1}$ of open neighborhoods Ω_i of ζ such that $\cap_{i \geq 1} \Omega_i = \{\zeta\}$.

18. Corollary to Lemma 15. *A point ζ in D_p^* $(1 \leq p \leq d)$ belongs to D if and only if ζ is G_δ.*

Proof: We only have to show that $\zeta \in \Gamma_p(D) = D_p^* \setminus D$ is not G_δ. Contrariwise suppose ζ is G_δ so that there exists a sequence $(\Omega_i)_{i \geq 1}$ of open neighborhoods of ζ such that $\Omega_i \supset \mathrm{cl}(\Omega_{i+1}; D_p^*)$ $(i = 1, 2, \cdots)$ and $\cap_{i \geq 1} \Omega_i = \{\zeta\}$. Since D is dense in D_p^*, $H_i := D \cap (\Omega_i \setminus \mathrm{cl}(\Omega_{i+1}; D_p^*))$ is a nonempty open subset of D for each i. Hence we can find a sequence $(P_n)_{n \geq 1}$ of points $P_n \in H_n$ $(n = 1, 2, \cdots)$ and a sequence $((U_n, x_n))_{n \geq 1}$ of 2-domains (U_n, x_n) contained in H_n $(n = 1, 2, \cdots)$ such that $U_n = \{x_n : |x_n - P_n| < r_n\}$ $(r_n > 0)$ $(n = 1, 2, \cdots)$. Let $R_n := \{x_n : a_n < |x_n - P_n| < b_n\}$ $(0 < a_n < b_n := r_n/2)$ be a spherical ring in (U_n, x_n). Clearly $(R_n)_{n \geq 1}$ is an admissible sequence. Since $\mathrm{cap}_p(R_n, \delta_{ij}) = \omega_d(|q|/(1 - (a_n/b_n)^{|q|}))^{p-1} a_n^{|d-p|}$ by (13) for $1 < p < d$, $\mathrm{cap}_d(R_n, \delta_{ij}) = \omega_d/(\log(b_n/a_n))^{d-1}$, and $\mathrm{cap}_1(R_n, \delta_{ij}) = \omega_d a_n^{d-1}$, we can see that $\mathrm{cap}_p(R_n, \delta_{ij}) < 2^{-n}$ by choosing $a_n \in (0, r_n/2)$ enough small so that $\mathrm{cap}_p R = \mathrm{cap}_p(R, g_{ij}) \leq 2^{(d+p)/2} \mathrm{cap}_p(R, \delta_{ij}) < 2^{(d+p)/2} 2^{-n}$ $(n = 1, 2, \cdots)$ by (11). Hence (17) is satisfied but (16) is invalid because the intersection on the left hand side of (16) contains ζ due to the fact that $R_n \subset H_n$ $(n = 1, 2, \cdots)$. This is clearly a contradiction to Lemma 15. □

19. Analytic properties of quasiisometric mappings. A *quasiisometric* (*quasiconformal*, resp.) mapping f of a Riemannian manifold D onto another D' is, as defined in §1 (Introduction), a homeomorphism f of D onto D' such that $K^{-1}\rho(x, y) \leq \rho(f(x), f(y)) \leq K\rho(x, y)$ for every pair of points x and y in D for some fixed $K \in [1, \infty)$ $(\sup_{x \in D}(\limsup_{r \downarrow 0}((\max_{\rho(x,y)=r} \rho'(f(x), f(y)))/(\min_{\rho(x,y)=r} \rho'(f(x), f(y)))) < \infty$, resp.), where ρ and ρ' are geodesic distances on D and D', respectively. In this case we also say that f is K-quasiisometric referring to K. For simplicity, quasiisometric (quasiconformal, resp.) mappings will occasionally be abbreviated as qi (qc, resp.). Consider a K-qi f of a d-

dimensional Riemannian manifold D equipped with the metric tensor (g_{ij}) onto another d-dimensional Riemannian manifold D' equipped with the metric tensor (g'_{ij}). Fix an arbitrary $\lambda \in (0,\infty)$ and choose any λ-domain (U,x) in D and any λ-domain (U',x') in D' such that $f(U)=U'$. The mapping $f:(U,\delta_{ij})\to(U',\delta_{ij})$ has the representation

$$x' = f(x) = (f^1(x),\cdots,f^d(x)) \tag{20}$$

on U in terms of the parameters x and x'. As the composite mapping of $id.:(U,\delta_{ij})\to(U,g_{ij})$, $f:(U,g_{ij})\to(U',g'_{ij})$, and $id.:(U',g'_{ij})\to(U',\delta_{ij})$, we see that the mapping $f:(U,\delta_{ij})\to(U',\delta_{ij})$ is λK-qi since $id.:(U,\delta_{ij})\to(U,g_{ij})$ and $id.:(U',g'_{ij})\to(U',\delta_{ij})$ are $\sqrt{\lambda}$-qi as the consequence of $\lambda^{-1}|dx|^2 \le ds^2 \le \lambda|dx|^2$, where $dx=(dx^1,\cdots,dx^d)$, $|dx|^2=\delta_{ij}dx^idx^j$, and $ds^2=g_{ij}(x)dx^idx^j$, which is deduced from $\lambda^{-1}(\delta_{ij})\le(g_{ij})\le\lambda(\delta_{ij})$. Hence we see that

$$\frac{1}{\lambda K}|x-y| \le |f(x)-f(y)| \le \lambda K|x-y| \tag{21}$$

whenever the line segment $[x,y]:=\{(1-t)x+ty : t\in[0,1]\}\subset U$ and $[f(x),f(y)]\subset U'$. In paticular (21) implies that

$$\limsup_{h\to 0}\frac{|f(x+h)-f(x)|}{|h|} \le \lambda K < \infty \tag{22}$$

for every $x\in U$ and

$$\liminf_{h\to 0}\frac{|f(x+h)-f(x)|}{|h|} \ge \frac{1}{\lambda K} > 0. \tag{23}$$

As an important consequence of (22), the Rademacher-Stepanoff theorem (cf. e.g. p.218 in [1]) assures that $f(x)$ is differentiable at a.e. $x\in U$, i.e.

$$f(x+h)-f(x) = f'(x)h+\varepsilon(x,h)|h| \qquad (\lim_{h\to 0}\varepsilon(x,h)=0) \tag{24}$$

for a.e. $x\in U$, where $f'(x)$ is the $d\times d$ matrix $(\partial f^i/\partial x^j)$. Fix an arbitrary vector h with $|h|=1$. Then for any positive number $t>0$ we have, by replacing h in(24) with th,

$$|f'(x)h| - |\varepsilon(x,th)| \le \frac{|f(x+th)-f(x)|}{|th|}$$

and on letting $t\downarrow 0$ we obtain by (22) that $|f'(x)h|\le\lambda K$. Therefore

$$|f'(x)| := \sup_{|h|=1}|f'(x)h| \le \lambda K \tag{25}$$

for a.e. $x\in U$. Similarly we have

$$|f'(x)h| + |\varepsilon(x,th)| \ge \frac{|f(x+th)-f(x)|}{|th|}$$

and hence by (23) we deduce $|f'(x)h| \geq 1/\lambda K$. Hence

$$l(f'(x)) := \inf_{|h|=1} |f'(x)h| \geq \frac{1}{\lambda K}. \tag{26}$$

From (25) it follows that $|\partial f^i(x)/\partial x^j| \leq |f'(x)| \leq \lambda K$ for a.e. $x \in U$ $(i, j = 1, \cdots, d)$ and thus $|\nabla f| = (\sum_{i=1}^d |\nabla f_i|^2)^{1/2} \in L^\infty(U)$. By (21), $f(x)$ is ACL (absolutely continuous on almost all straight lines which are parallel to coordinate axes). That $f(x)$ is ACL and $\nabla f \in L^\infty(U)$ is necessary and sufficient for f to belong to $L^{1,\infty}(U)$ (cf. e.g. pp.8-9 in [7]) so that, by the continuity of f, we have

$$f \in W^{1,\infty}_{loc}(D). \tag{27}$$

By (25) and (26) we have the matrix inequality

$$l(f'(x))^2(\delta_{ij}) \leq f'(x)^* f'(x) \leq |f'(x)|^2(\delta_{ij})$$

for a.e. $x \in U$, where $f'(x)^*$ is the transposed matrix of $f'(x)$. Let $\lambda_1(x) \geq \cdots \geq \lambda_d(x)$ be the square roots of the proper values of the symmetric positive matrix $f'(x)^* f'(x)$. Then

$$\frac{1}{\lambda K} \leq l(f'(x)) = \lambda_d(x) \leq \cdots \leq \lambda_1(x) = |f'(x)| \leq \lambda K.$$

Observe that $\prod_{i=1}^d \lambda_i(x)^2 = \det(f'(x)^* f'(x)) = (\det f'(x))^2$ is the square of the Jacobian $J_f(x)$ of f at x. Hence, by $\lambda K \lambda_i \geq 1$ $(i = 2, 3, \cdots, d)$, we see that

$$\begin{aligned} |f'(x)|^p = \lambda_1(x)^p \leq \lambda_1(x)(\lambda K)^{p-1} &\leq \lambda_1(x)(\lambda K)^{p-1} \prod_{i=2}^d (\lambda K \lambda_i(x)) \\ &= (\lambda K)^{d+p-2} \prod_{i=1}^d \lambda_i(x) = (\lambda K)^{d+p-2} |J_f(x)|, \end{aligned}$$

i.e. we have deduced that

$$|f'(x)|^p \leq (\lambda K)^{d+p-2} |J_f(x)| \tag{28}$$

for a.e. $x \in U$. This is used to prove the following result.

29. PROPOSITION. *The pull-back $v = u \circ f$ of any u in $M_p(D')$ by a K-qi f of D onto D' belongs to $M_p(D)$ and satisfies the inequality*

$$\int_D |\nabla v(x)|^p_{g_{ij}} \sqrt{g(x)} dx \leq K^{d+p-2} \int_{D'} |\nabla u(x')|^p_{g'_{ij}} \sqrt{g'(x')} dx' \tag{30}$$

and in particular

$$\|v; M_p(D)\| \leq K^{(d+p-2)/p} \|u; M_p(D')\|. \tag{31}$$

PROOF: The inequality (30) is nothing but $\|\nabla v; L^p(D)\| \le K^{(d+p-2)/p}\|\nabla u; L^p(D')\|$. This with $\|v; L^\infty(D)\| = \|u; L^\infty(D')\|$ implies (31). Suppose that Proposition 29 is true if $u \in M_p(D') \cap C^\infty(D')$. Since $M_p(D') \cap C^\infty(D')$ is dense in $M_p(D')$, for an arbitrary $u \in M_p(D')$, there exists a sequence $(u_k)_{k\ge1}$ in $M_p(D') \cap C^\infty(D')$ such that $\|u - u_k; M_p(D')\| \to 0$ $(k \to \infty)$. In particular $\|u_k - u_{k'}; M_p(D')\| \to 0$ $(k, k' \to \infty)$. By our assumption, $v_k := u_k \circ f \in M_p(D)$ $(k = 1, 2, \cdots)$. By (31), the inequalities $\|v_k - v_{k'}; M_p(D)\| \le K^{(d+p-2)/p}\|u_k - u_{k'}; M_p(D')\|$ assure that $\|v_k - v_{k'}; M_p(D)\| \to 0$ $(k, k' \to \infty)$. By the completeness of $M_p(D)$, since $\|v - v_k; L^\infty(D)\| \to 0$ $(k \to \infty)$, we see that $v \in M_p(D)$. By the validity of (30) (and hence of (31)) for v_k, we see that (30) is valid for v. For this reason we can assume $u \in M_p(D') \cap C^\infty(D')$ to prove Proposition 29.

It is clear by (25) that $v = u \circ f \in W^{1,\infty}_{loc} \cap L^\infty(D) \cap C(D)$ if $u \in M_p(D') \cap C^\infty(D')$. Hence we only have to prove (30) to deduce $v \in M_p(D)$. Fix an arbitrary $\lambda \in (1, \infty)$. Let $D = \cup_{k=1}^\infty E_k$ be a union of disjoint Borel sets E_k in D such that each E_k is contained in a λ-domain U_k in D and $E_k' = f(E_k)$ in a λ-domain $U_k' = f(U_k)$ in D' for $k = 1, 2, \cdots$. Fix a k and consider the λK-qi f of (U_k, δ_{ij}) onto (U_k', δ_{ij}) with the representation (20) on U_k in terms of the parameter x in U_k and x' in U_k'. By the chain rule we have

$$\nabla v(x) = f'(x)^* \nabla u(f(x)) \tag{32}$$

for a.e. $x \in U_k$. Since $|f'(x)^*| = |f'(x)|$, (28) and (32) yield

$$|\nabla v(x)|^p \le (\lambda K)^{d+p-2} |\nabla u(f(x))|^p |J_f(x)|$$

for a.e. $x \in U_k$. In view of (22), the formula of the change of variables in integrations is valid for $x' = f(x)$:

$$\int_{E_k} |\nabla u(f(x))|^p |J_f(x)| dx = \int_{E_k'} |\nabla u(x')|^p dx'.$$

From the above two displayed relations we deduce

$$\int_{E_k} |\nabla v(x)|^p dx \le (\lambda K)^{d+p-2} \int_{E_k'} |\nabla u(x')|^p dx'.$$

Observe that $|\nabla v|^p_{g_{ij}} \le \lambda^{p/2} |\nabla v|^p$ and $\sqrt{g} \le \lambda^{d/2}$, and simlarly, that $|\nabla u|^p \le \lambda^{p/2} |\nabla u|^p_{g'_{ij}}$ and $1 \le \lambda^{d/2}\sqrt{g'}$. The above displayed inequality then implies that

$$\int_{E_k} |\nabla v(x)|^p_{g_{ij}} \sqrt{g(x)} dx \le \lambda^{2(d+p-1)} K^{d+p-2} \int_{E_k'} |\nabla u(x')|^p \sqrt{g'(x')} dx'.$$

On adding these inequalities for $k = 1, 2, \cdots$ we obtain (30) with K^{d+p-2} replaced by $\lambda^{2(p+d-1)} K^{d+p-2}$. Since $\lambda \in (1, \infty)$ is arbitrary, we deduce (30) itself by letting $\lambda \downarrow 1$. □

33. Distortion of rings and their capacities. Throughout this section we fix two nonempty open sets V and V' in $\mathbf{R}^d$ (or, what amounts to the same, two parametric domains

(V, x) and (V', x') in certain Riemannian manifolds D and D', respectively, considered as (V, δ_{ij}) and (V', δ_{ij})) and consider homeomorphisms f of V onto V'. We introduce two classes of such homeomorphisms f. The first class $Lip(K) = Lip(K; V, V')$ for a positive constant $K \in (0, \infty)$ is the family of homeomorphisms f of V onto V' such that

$$\limsup_{r \downarrow 0} \frac{\max_{|x-P|=r} |f(x) - f(P)|}{r} \leq K \tag{34}$$

at every point $P \in V$. If the inverse f^{-1} of a homeomorphism f of V onto V' satisfies the similar property as (34), then we should write $f^{-1} \in Lip(K; V', V)$ but we often loosely write $f^{-1} \in Lip(K)$. This class was first introduced by Gehring [3]. Note that $f(R)$ may be viewed as a ring in V' in the natural fashion along with a ring R in V: the inner part and the outer part of $f(R)^c = V' \setminus f(R)$ are the images of those of $R^c = V \setminus R$ under f, respectively. For each $p \in (1, \infty)$ the second class $Q_p(K, \delta) = Q_p(K, \delta; V, V')$ for two constants $K \in (0, \infty)$ and $\delta \in (0, \infty]$ is defined to be the family of homeomorphisms f of V onto V' satisfying the following condition:

$$\mathrm{cap}_p f(R) \leq K \mathrm{cap}_p R \tag{35}$$

for every *spherical* ring R in V such that $\overline{R} \subset V$ and

$$\mathrm{cap}_p R < \delta. \tag{36}$$

In the case $\delta = \infty$ the condition (36) is redundant and thus the condition is given only by (35). The same remark as for the use of notation $f^{-1} \in Lip(K)$ also applies to the use of $f^{-1} \in Q_p(K, \delta)$. Clearly we see that $Q_p(K, \infty) \subset Q_p(K, \delta) \subset Q_p(K', \delta')$ for $0 < K \leq K' < \infty$ and $0 < \delta' \leq \delta \leq \infty$. The class $Q_p(K, \infty)$ was introduced by Gehring [3] under the notation $Q_p(K)$. The following result plays a key role in the proof of our main theorem 4 in this paper.

37. LEMMA. *Suppose that $1 \leq p < d$, $0 < K < \infty$, and $0 < \delta \leq \infty$ are arbitrarily given. Then $f, f^{-1} \in Q_p(K, \delta)$ implies that $f, f^{-1} \in Lip(K_1)$, where K_1 depends only upon d, p, and K and does not depend on δ. Explicitly, K_1 can be chosen as*

$$K_1 = K_1(K) := K^{\frac{1}{d-p}} \exp\left(\left(2^{d+1} \omega_d^{1+\frac{1}{d}} K^{\frac{2(d-1)}{d-p}} t_d^{-\frac{1}{d}}\right)^{\frac{d}{d-1}}\right). \tag{38}$$

Recall that t_d was given in (14). This lemma 37 is partly a generalization of the Gehring theorem ([3]): $f, f^{-1} \in Q_p(K, \infty)$ for $1 \leq p < \infty$ with $p \neq d$ and $0 < K < \infty$ implies $f, f^{-1} \in Lip(K')$, where K' depends only upon d, p, and K. Namely, Lemma 37 contains the Gehring theorem for $1 \leq p < d$. However Lemma 37 is no longer true especially for small finite positive numbers $\delta > 0$ if $1 \leq p < d$ is replaced by $d < p \leq \infty$. Nevertheless, Lemma 37 can be proven by suitably modifying the original Gehring proof ([3]) of his theorem. A complete proof of Lemma 37 can be found in [12].

If we assume that f is K_1-qi, then $f, f^{-1} \in Lip(K_1)$, which is the conclusion of Lemma 37, follows immediately. We now prove the converse of this so that $f, f^{-1} \in Lip(K)$ can be used for the definition of K-qi in the case of mappings between space open sets.

39. LEMMA. *If $f, f^{-1} \in Lip(K)$, then f is a K-qi of V onto V'.*

PROOF: We define positive numbers $s(r) > 0$ for sufficiently small positive numbers $r > 0$ by $\min_{|x-P|=r} |f(x) - f(P)| =: s(r)$ for an arbitrarily fixed $P \in V$. On setting $P' := f(P)$ we see that $\max_{|x'-P'|=s(r)} |f^{-1}(x') - f^{-1}(P')| = r$. Observe that $s(r) \downarrow 0$ as $r \downarrow 0$. Hence, by $f^{-1} \in Lip(K) = Lip(K; V', V)$, we see that

$$\begin{aligned}\limsup_{r\downarrow 0} \frac{r}{s(r)} &= \limsup_{r\downarrow 0} \frac{\max\limits_{|x'-P'|=s(r)} |f^{-1}(x') - f^{-1}(P')|}{s(r)} \\ &\leq \limsup_{s\downarrow 0} \frac{\max\limits_{|x'-P'|=s} |f^{-1}(x') - f^{-1}(P')|}{s} \leq K.\end{aligned}$$

Therefore we infer that

$$\begin{aligned}\limsup_{r\downarrow 0} \frac{\max\limits_{|x-P|=r} |f(x) - f(P)|}{\min\limits_{|x-P|=r} |f(x) - f(P)|} &= \limsup_{r\downarrow 0} \left(\frac{\max\limits_{|x-P|=r} |f(x) - f(P)|}{r} \cdot \frac{r}{s(r)} \right) \\ &\leq \left(\limsup_{r\downarrow 0} \frac{\max\limits_{|x-P|=r} |f(x) - f(P)|}{r} \right) \cdot \left(\limsup_{r\downarrow 0} \frac{r}{s(r)} \right) \leq K^2,\end{aligned}$$

which concludes that f is a qc of V onto V' by the metric definition (2) of quasiconformality. This assures that f is differentiable a.e. on V and $f \in W^{1,d}_{loc}(V)$ (cf. e.g. pp.109-111 in [19]). The latter in particular implies that f is ACL in an arbitrarily given direction l: f is absolutely continuous on almost all straight lines which are parallel to l. Suppose that f is differentiable at $x \in V$, i.e.

$$f(x+h) - f(x) = f'(x)h + \varepsilon(x,h)|h| \quad (\lim_{h\to 0} \varepsilon(x,h) = 0).$$

For any $|h| = 1$ and any small $t > 0$, we have

$$|f'(x)h| \leq \frac{|f(x+th) - f(x)|}{|th|} + |\varepsilon(x,th)| \leq \frac{\max\limits_{|y-x|=t} |f(y) - f(x)|}{t} + |\varepsilon(x,th)|.$$

On letting $t \downarrow 0$ we deduce $|f'(x)h| \leq K$ since $f \in Lip(K)$. We can thus conclude that

$$|f'(x)| = \sup_{|h|=1} |f'(x)h| \leq K \tag{40}$$

for a.e. $x \in U$. We now maintain that

$$(41) \qquad |f(x) - f(y)| \leq K|x - y|$$

for any line segment $[x, y] = \{(1-t)x + ty : t \in [0, 1]\} \subset V$. Since f is ACL in the direction of $[x, y]$, we see that f is absolutely continuous in V on almost all straight lines L parallel to $[x, y]$. As a consequence of (40), $|f'(x)| \leq K$ in V on almost all straight lines L parallel to $[x, y]$ a.e. with respect to the linear measure on L. Hence we can find a sequence of line segments $[x_n, y_n] \subset V$ with the following properties: $x_n \to x$ and $y_n \to y$ as $n \to \infty$; f is absolutely continuous on $[x_n, y_n]$; $|f'(x)| \leq K$ a.e. on $[x_n, y_n]$ with respect to the linear measure. Then

$$\begin{aligned} |f(x_n) - f(y_n)| &\leq \int_{[x_n, y_n]} |df(z)| = \int_{[x_n, y_n]} |f'(z)dz| \\ &\leq \int_{[x_n, y_n]} |f'(z)||dz| \leq K \int_{[x_n, y_n]} |dz| = K|x_n - y_n|, \end{aligned}$$

i.e. $|f(x_n) - f(y_n)| \leq K|x_n - y_n|$ $(n = 1, 2, \cdots)$, from which (41) follows by the continuity of f. By the symmetry of the situations for f and f^{-1}, we deduce the same inequality for f^{-1} so that

$$\frac{1}{K}|x - y| \leq |f(x) - f(y)| \leq K|x - y|$$

for every x and y in V with $[x, y] \subset V$ and $[f(x), f(y)] \subset V'$. Thus we can show the validity of (3) with respect to δ_{ij}-geodesic distances ρ on V and ρ' on V' so that $f : V \to V'$ is a K-qi. □

Combining Lemmas 37 and 39, we obtain the following result, which will be used in the final part of the proof of the main theorem 4.

42. Theorem. *Suppose that* $1 \leq p < d$, $0 < K < \infty$, *and* $0 < \delta \leq \infty$ *are arbitrarily given. Then* $f, f^{-1} \in Q_p(K, \delta)$ *implies that* f *is a* K_1*-qi of* V *onto* V', *where* $K_1 = K_1(K)$ *is given by* (38) *so that it is independent of* δ.

43. Proof of the main theorem. In this section we assume that the exponent p is fixed in $(1, d)$ and we choose two Riemannian manifolds D and D' of the same dimension $d \geq 2$ which are orientable and countable and any component of D and D' is not compact. The proof of the main theorem 4 consits of two parts.

First part: Assume that there exists an almost quasiisometric mapping f of D onto D', i.e. f is a homeomorphism of D onto D' and there exists a compact subset $E \subset D$ such that $f = f|D \setminus E$ is a K-quasiisometric mapping of $D \setminus E$ onto $D' \setminus E'$, where $E' = f(E)$ is a compact subset of D' and K a constant in $[1, \infty)$. We are to show that f can be extended to a homeomorphism f^* of the Royden compactification D_p^* of D onto that $(D')_p^*$ of D'. Choose an arbitrary point ξ in the Royden p-boundary $\Gamma_p(D) = D_p^* \setminus D$. Since D is dense in D_p^*, the point ξ is an accumulation point of D.

We first show that the net $(f(x_\lambda))$ in D' converges to a point $\xi' \in \Gamma_p(D')$ for any net (x_λ) in D convergent to ξ. Clearly $(f(x_\lambda))$ does not accumulate at any point in D' along with (x_λ) so that the cluster points of $(f(x_\lambda))$ are contained in $\Gamma_p(D')$. Contrariwise we assume the existence of two subnets $(x_{\lambda'})$ and $(x_{\lambda''})$ of (x_λ) such that $(f(x_{\lambda'}))$ and $(f(x_{\lambda''}))$ are convergent to η' and η'' in $\Gamma_p(D')$, respectively, with $\eta' \neq \eta''$. Since $M_p(D')$ is dense in $C((D')^*_p)$ and forms a lattice, we can find a function $u \in M_p(D')$ such that $u \equiv 0$ in a neighborhood G' of E', $u(\eta') = 0$, and $u(\eta'') = 1$. Viewing $u \in M_p(D' \setminus E')$, we see by Proposition 29 that $v := u \circ f \in M_p(D \setminus E)$. Since $v \equiv 0$ on the neighborhood $G = f^{-1}(G')$ of $E = f^{-1}(E')$, we can conclude that $v \in M_p(D)$. From $v(x_{\lambda'}) = u(f(x_{\lambda'}))$ and $v(x_{\lambda''}) = u(f(x_{\lambda''}))$ it follows that $v(\xi) = u(\eta') = 0$ and $v(\xi) = u(\eta'') = 1$, which is a contradiction.

We next show that the nets $(f(x_{\lambda'}))$ and $(f(y_{\lambda''}))$ in D' converge to a point in $\Gamma_p(D')$ for any two nets $(x_{\lambda'})$ and $(y_{\lambda''})$ convergent to $\xi \in \Gamma_p(D)$. In fact, let (z_λ) be a net convergent to ξ such that (z_λ) contains $(x_{\lambda'})$ and $(y_{\lambda''})$ as its subnets. Then we see that $\lim_\lambda f(x_{\lambda'}) = \lim_{\lambda''} f(y_{\lambda''}) = \lim_\lambda f(z_\lambda)$. Hence we have shown that $f^*(\xi) := \lim_{x \in D, x \to \xi} f(x) \in \Gamma_p(D')$ for any $\xi \in \Gamma_p(D)$. On setting $f^* = f$ on D, we see that f^* is a continuous mapping of D^*_p onto $(D')^*_p$. The uniqueness of f^* on D^*_p is a consequence of the denseness of D in D^*_p. Similarly we can show that f^{-1} can also be uniquely extended to a continuous mapping $(f^{-1})^*$ of $(D')^*_p$ onto D^*_p. Since $(f^{-1})^* \circ f^*$ and $f^* \circ (f^{-1})^*$ are identities on D^*_p and $(D')^*_p$, respectively, as the unique extensions of *id.* : $D \to D$ and *id.* : $D' \to D'$, respectively, we see that f^* is a homeomorphism of D^*_p onto $(D')^*_p$ which is the unique extension of $f : D \to D'$. □

Second part : Suppose the existence of a homeomorphism f^* of D^*_p onto $(D')^*_p$. We are to show that $f := f^*|D$ is an almost quasiisometric mapping of D onto D', which is the essential part of this note.

Choose an arbitrary point $x \in D$. Since x is G_δ, $f^*(x) \in (D')^*_p$ is also G_δ so that $f^*(x) \in D'$ by Corollary 18. Thus we have shown that $f^*(D) \subset D'$. Similarly we can conclude that $(f^*)^{-1}(D') \subset D$. These show that $f^*(D) = D'$ and therefore $f := f^*|D$ is a homeomorphism of D onto D'. We are to show that f is an almost quasiisometric mapping of D onto D'.

We fix a family $\mathcal{V} = \mathcal{V}_D = \{V\}$ of open sets V in D with the following properties: V is contained in a 2-domain U_V in D and $V' := f(V)$ is contained in the 2-domain $U'_{V'} = f(U_V)$ in D'; $\cup_{V \in \mathcal{V}} V = D$. This is possible since the family of 2-domains forms a base of open sets on any Riemannian manifold and $f : D \to D'$ is a homeomorphism. We set $\mathcal{V}' := \{V' : V' = f(V) \ \ (V \in \mathcal{V})\}$, which enjoys the same properties as $\mathcal{V}$ does. We also fix an exhaustion $(\Omega_n)_{n \geq 1}$ of D, i.e. Ω_n is a relatively compact open subset of D $(n = 1, 2, \cdots)$, $\overline{\Omega_n} \subset \Omega_{n+1}$ $(n = 1, 2, \cdots)$, and $\cup_{n \geq 1} \Omega_n = D$. Then $(\Omega'_n)_{n \geq 1}$ with $\Omega'_n := f(\Omega_n)$ $(n = 1, 2, \cdots)$ also forms an exhaustion of D'. We set $D_n := D \setminus \overline{\Omega_n}$ and $D'_n := f(D_n) = D' \setminus \overline{\Omega'_n}$ $(n = 1, 2, \cdots)$. Then $(D_n)_{n \geq 1}$ $((D'_n)_{n \geq 1}$, resp.) is a decreasing sequence of open sets D_n

(D'_n, resp.) with compact complements $D \setminus D_n$ ($D' \setminus D'_n$, resp.) such that $\cap_{n\geq 1} D_n = \emptyset$ ($\cap_{n\geq 1} D'_n = \emptyset$, resp.). If we set $\mathcal{V}_{D_n} := \{V \cap D_n : V \in \mathcal{V}_D \text{ and } V \cap D_n \neq \emptyset\}$ $(n = 1, 2, \cdots)$, then $\mathcal{V}_{D_n}$ plays the same role for D_n as $\mathcal{V}$ does for D. Take an arbitrary $n \in \{1, 2, \cdots\}$. If $f \in Q_p(2^{n+p-1}, 2^{-n}; V \cap D_n, V' \cap D'_n)$ ($f^{-1} \in Q_p(2^{n+p-1}, 2^{-n}; V' \cap D'_n, V \cap D_n)$, resp.) for every $V \in \mathcal{V}$ with $V \cap D_n \neq \emptyset$ (so that $V' \cap D'_n \neq \emptyset$), where $V' = f(V)$ and $V' \cap D'_n = f(V) \cap f(D_n) = f(V \cap D_n)$, then we write

$$f \in (n) \qquad (f^{-1} \in (n), \text{ resp.}).$$

Hence, for example, $f \notin (n)$ means that there exists a $V \in \mathcal{V}$ with $V \cap D_n \neq \emptyset$ such that $f \notin Q_p(2^{n+p-1}, 2^{-n}; V \cap D_n, V' \cap D'_n)$. We maintain

44. Assertion. *If $f \in (n)$ ($f^{-1} \in (n)$, resp.) for some n, then $f \in (m)$ ($f^{-1} \in (m)$, resp.) for every $m \geq n$.*

In fact, $f \in (n)$ assures that $f \in Q_p(2^{n+p-1}, 2^{-n}; V \cap D_n, V' \cap D'_n)$ for every $V \in \mathcal{V}$ with $V \cap D_n \neq \emptyset$. Choose any $V \in \mathcal{V}$ with $V \cap D_m \neq \emptyset$. Since $D_m \subset D_n$, $V \cap D_n \neq \emptyset$ along with $V \cap D_m \neq \emptyset$ and therefore $f \in Q_p(2^{n+p-1}, 2^{-n}; V \cap D_n, V' \cap D'_n)$. In view of the fact that $2^{n+p-1} \leq 2^{m+p-1}$ and $2^{-n} \geq 2^{-m}$, we have the inclusion relation $Q_p(2^{m+p-1}, 2^{-m}; V \cap D_m, V' \cap D'_m) \supset Q_p(2^{n+p-1}, 2^{-n}; V \cap D_n, V' \cap D'_n)$ so that $f \in Q_p(2^{m+p-1}, 2^{-m}; V \cap D_m, V' \cap D'_m)$, i.e. $f \in (m)$, which completes the proof of Assertion 44. Next we assert

45. Assertion. *If $f \in (n)$ and $f^{-1} \in (n)$ for some n, then $f = f|D_n$ is a qi of D_n onto D'_n.*

Indeed, by Theorem 42, we see that $f : (V \cap D_n, \delta_{ij}) \to (V' \cap D'_n, \delta_{ij})$ is a K_1-qi with $K_1 = K_1(2^{n+p-1})$ (cf. (38) in Lemma 37). Clearly *id.* : $(V \cap D_n, g_{ij}) \to (V \cap D_n, \delta_{ij})$ and *id.* : $(V' \cap D'_n, \delta_{ij}) \to (V' \cap D'_n, g'_{ij})$ are $\sqrt{2}$-qi, where (g'_{ij}) is the metric tensor on D'. Therefore, as the suitable composition of these maps above, we see that $f : (V \cap D_n, g_{ij}) \to (V' \cap D'_n, g'_{ij})$ is a $2K_1$-qi. Since this is true for every $V \in \mathcal{V}$ with $V \cap D_n \neq \emptyset$ and $\cup_{V\in\mathcal{V}} V = D \supset D_n$, we can conclude that $f : D_n \to D'_n$ is a $2K_1$-qi. The proof of Assertion 45 is thus complete.

To complete the proof of this second part it is sufficient to show that $f : D_n \to D'_n$ is a qi for some n. We prove it by contradiction. Contrariwise suppose that $f : D_n \to D'_n$ is not qi for every $n = 1, 2, \cdots$. Then we maintain that either $f \notin (n)$ for every n or $f^{-1} \notin (n)$ for every n. In fact, if $f \notin (n)$ for every n, then we are done. Otherwise, there is a k with $f \in (k)$. Then by Assertion 44 we have $f \in (n)$ for every $n \geq k$. In this case we must have $f^{-1} \notin (n)$ for every n and the assertion is assured. To see this assume that $f^{-1} \in (l)$ for some l. Then $f^{-1} \in (n)$ for every $n \geq l$ again by Assertion 44. Then $f \in (k \cup l)$ and $f^{-1} \in (k \cup l)$. By Assertion 45 we see that f is a qi of $D_{k\cup l}$ onto $D'_{k\cup l}$, contradicting our assumption. On interchanging the roles of f and f^{-1} (and thus those of D and D') if necessary, we can assume that

$$f \notin (n) \qquad (n = 1, 2, \cdots),$$

from which we will derive a contradiction.

The fact that $f \notin (1)$ implies the existence of a 2-domain $V \in \mathcal{V}_{D_1}$ such that $f \notin Q_p(2^{1+p-1}, 2^{-1}; V, f(V))$. We can then find a spherical ring $S_1 \subset V(\subset D_1)$ such that

$$\mathrm{cap}_p S_1 < 2^{-1}, \qquad \mathrm{cap}_p f(S_1) > 2^{1+p-1}\mathrm{cap}_p S_1.$$

Here $\mathrm{cap}_p S_1$ means $\mathrm{cap}_p(S_1, \delta_{ij})$. We set $n_1 := 1$. Let n_2 be the least integer such that $n_2 \geq n_1 + 1$ (and hence $D_{n_1+1} \supset D_{n_2}$) and $\overline{D_{n_2}} \cap \overline{S_{n_1}} = \emptyset$. Since $f \notin (n_2)$, there exists a $V \in \mathcal{V}_{D_{n_2}}$ with $f \notin Q_p(2^{n_2+p-1}, 2^{-n_2}; V, f(V))$. Hence we can find a spherical ring $S_{n_2} \subset V(\subset D_{n_2})$ such that

$$\mathrm{cap}_p S_{n_2} < 2^{-n_2}, \qquad \mathrm{cap}_p f(S_{n_2}) > 2^{n_2+p-1}\mathrm{cap}_p S_{n_2},$$

where $\mathrm{cap}_p S_{n_2}$ means $\mathrm{cap}_p(S_{n_2}, \delta_{ij})$. Repeating this process we can construct a sequence $(S_{n_k})_{k\geq 1}$ of spherical rings S_{n_k} with the following properties: $n_k + 1 \leq n_{k+1}$; $S_{n_k} \subset D_{n_k}$; $\overline{S_{n_k}} \cap \overline{S_{n_l}} = \emptyset$ $(k \neq l)$;

$$\mathrm{cap}_p S_{n_k} < 2^{-n_k}, \quad \mathrm{cap}_p f(S_{n_k}) > 2^{n_k+p-1}\mathrm{cap}_p S_{n_k} \quad (k = 1, 2, \cdots). \tag{46}$$

Fix a k and set $T = S_{n_k}$. Since it is a spherical ring in a 2-domain $(U_{V_{n_k}}, x)$ and contained in V_{n_k}, T has a representation $T = \{x : a < |x - P| < b\}$, where $P \in V_{n_k}$ and $0 < a < b < \infty$. Let $l = [(2^{-n_k}/\mathrm{cap}_p T)^{1/(p-1)}] > 0$, where [ ] is the Gaussian symbol, which means that

$$l^{p-1} \leq \frac{2^{-n_k}}{\mathrm{cap}_p T} < (l+1)^{p-1} \leq 2^{p-1} l^{p-1}. \tag{47}$$

Using the notation $q = (p-d)/(p-1)$ (cf. (13)) we set

$$t_j := \left(\frac{(l-j)a^q + jb^q}{l}\right)^{\frac{1}{q}} \qquad (j = 0, 1, \cdots, l).$$

We divide the ring T into l small sphherical rings T_j given by

$$T_j := \{x : t_{j-1} < |x - P| < t_j\} \qquad (j = 0, 1, \cdots, l).$$

By (13) we have $\mathrm{cap}_p T = \mathrm{cap}_p(T, \delta_{ij}) = \omega_d((b^q - a^q)/q)^{1-p}$. Similarly

$$\mathrm{cap}_p T_j = \omega_d \left(\frac{t_j^q - t_{j-1}^q}{q}\right)^{1-p}$$

$$= \omega_d \left(\frac{\frac{(l-j)a^q + jb^q}{l} - \frac{(l-j+1)a^q + (j-1)b^q}{l}}{q} \right)^{1-p}$$
$$= \omega_d \left(\frac{b^q - a^q}{q} \right) l^{p-1} = l^{p-1}\mathrm{cap}_p T,$$

i.e. we have shown that $\mathrm{cap}_p T_j = l^{p-1}\mathrm{cap}_p T$. Therefore we have the following identity for the subdivision $\{T_j\}_{1\le j\le l}$ of T:

$$\sum_{j=1}^{l} (\mathrm{cap}_p T_j)^{\frac{1}{1-p}} = (\mathrm{cap}_p T)^{\frac{1}{1-p}}. \tag{48}$$

Concerning the induced subdivision $\{f(T_j)\}$ of $f(T)$, the general inequality (10) implies the inequality

$$\sum_{j=1}^{l} (\mathrm{cap}_p f(T_j))^{\frac{1}{1-p}} \le (\mathrm{cap}_p f(T))^{\frac{1}{1-p}}. \tag{49}$$

Now suppose that $\mathrm{cap}_p f(T_j) \le 2^{n_k+p-1}\mathrm{cap}_p T_j$ for every $1 \le j \le l$. Then $(\mathrm{cap}_p f(T_j))^{1/(1-p)} \ge 2^{(n_k+p-1)/(1-p)}(\mathrm{cap}_p T_j)^{1/(1-p)}$ for every $1 \le j \le l$. By using (49) and (48) we deduce

$$(\mathrm{cap}_p f(T))^{\frac{1}{1-p}} \ge \sum_{j=1}^{l} (\mathrm{cap}_p f(T_j))^{\frac{1}{1-p}}$$
$$\ge 2^{\frac{n_k+p-1}{1-p}} \sum_{j=1}^{l} (\mathrm{cap}_p T_j)^{\frac{1}{1-p}} = 2^{\frac{n_k+p-1}{1-p}} (\mathrm{cap}_p T)^{\frac{1}{1-p}},$$

which means that $\mathrm{cap}_p f(T) \le 2^{n_k+p-1}\mathrm{cap}_p T$. This contradicts (46) since $T = S_{n_k}$. Therefore there must exist a number $j_0 \in \{1, \cdots, l\}$ such that

$$\mathrm{cap}_p f(T_{j_0}) > 2^{n_k+p-1}\mathrm{cap}_p T_{j_0}. \tag{50}$$

We now set $R_k := T_{j_0}$. By (47) we have $l^{p-1}\mathrm{cap}_p T \le 2^{-n_k} \le 2^{p-1}l^{p-1}\mathrm{cap}_p T$. Since $l^{p-1}\mathrm{cap}_p T = \mathrm{cap}_p T_{j_0} = \mathrm{cap}_p R_k$, we see that

$$\mathrm{cap}_p R_k \le 2^{-n_k} \le 2^{p-1}\mathrm{cap}_p R_k.$$

This is equivalent to $\mathrm{cap}_p R_k \le 2^{-n_k} (< 2^{-k}$ (since $n_k > k$)) and $\mathrm{cap}_p R_k \ge 2^{-n_k-p+1}$. The latter inequality with (50) implies that $\mathrm{cap}_p f(R_k) > 2^{n_k+p-1}\mathrm{cap}_p R_k \ge 2^{n_k+p-1} \cdot 2^{-n_k-p+1} = 1$. By (46), $\mathrm{cap}_p(R_k, g_{ij}) < 2^{(d+p)/2} \cdot 2^{-k}$ and $\mathrm{cap}_p(f(R_k), g_{ij}) > 2^{(d+p)/2}$.

We have thus constructed an admissible sequence $(R_k)_{k\ge1}$ of rings R_k in D in the sense of §8 (cf. Lemma 15) such that $\mathrm{cap}_p R_k = \mathrm{cap}_p(R_k, g_{ij})$ and $\mathrm{cap}_p f(R_k) = \mathrm{cap}_p(f(R_k), g'_{ij})$ satisfy

$$\mathrm{cap}_p R_k < 2^{(d+p)/2} \cdot 2^{-k} \quad \text{and} \quad \mathrm{cap}_p f(R_k) > 2^{(d+p)/2} \tag{51}$$

for every $k = 1, 2, \cdots$. Let C_{k1} be the inner part of $R_k^c = D \setminus R_k$ and we set

$$X := \bigcup_{k=1}^{\infty} C_{k1} \qquad \text{and} \qquad Y := \bigcap_{k=1}^{\infty} (D \setminus (R_k \cup C_{k1}))$$

as in §8 (cf. Lemma 15). The first inequality in (51) implies that

$$\sum_{k=1}^{\infty} \mathrm{cap}_p R_k < \sum_{k=1}^{\infty} 2^{\frac{d+p}{2}} \cdot 2^{-k} = 2^{\frac{d+p}{2}} < \infty$$

and therefore Lemma 15 assures that

$$(\mathrm{cl}(X; D_p^*)) \cap (\mathrm{cl}(Y; D_p^*)) = \emptyset.$$

Due to the fact that f^* is a homeomorphism of D_p^* onto $(D')_p^*$, we see that

$$\begin{aligned}(\mathrm{cl}(f(X); (D')_p^*)) \cap (\mathrm{cl}(f(Y); (D')_p^*)) &= f^*(\mathrm{cl}(X, D_p^*)) \cap f^*(\mathrm{cl}(Y; D_p^*)) \\ &= f^*((\mathrm{cl}(X; D_p^*)) \cap (\mathrm{cl}(Y; D_p^*))) = f^*(\emptyset) = \emptyset.\end{aligned}$$

Since again $(f(R_k))_{k \geq 1}$ is an admissible sequence of rings $f(R_k)$ on D', the above relation must imply by Lemma 15 that $\sum_{k=1}^{\infty} \mathrm{cap}_p f(R_k) < \infty$. However the second inequality in (51) implies that

$$\sum_{k=1}^{\infty} \mathrm{cap}_p f(R_k) \geq \sum_{k=1}^{\infty} 2^{\frac{d+p}{2}} = \infty,$$

which is a contradiction. This comes from the erroneous assumption that $f : D_n \to D'_n$ is not a qi for every $n = 1, 2, \cdots$, and thus we have established the existence of an n such that $f = f|D_n$ is a qi of D_n onto D'_n. The second part of the proof for the main theorem 4 is herewith complete. □

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